CO-ORDINATE GEOMETRY

Books by Charles Smith

AN ELEMENTARY TREATISE ON SOLID GEOMETRY

ELEMENTARY ALGEBRA

A TREATISE ON ALGEBRA

CO-ORDINATE GEOMETRY

BY

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MACMILLAN & CO LTD
NEW YORK · ST MARTIN'S PRESS

First Edition printed 1882. *Second Edition printed* 1883. *Reprinted with slight corrections* 1884, *with slight corrections* 1885, 1887, 1888, 1889, *January and November* 1890, 1892, 1893, 1894, 1896, 1898, 1899, 1901, 1902, 1904, 1905, 1906, 1908, 1909. *Revised and enlarged* 1910. *Reprinted* 1912, 1914, 1916, 1918, 1919, 1920, 1921, 1924, 1927, 1930, 1937, 1943, 1946, 1953, 1956, 1962.

MACMILLAN AND COMPANY LIMITED
St Martin's Street London WC2
also Bombay Calcutta Madras Melbourne

THE MACMILLAN COMPANY OF CANADA LIMITED
Toronto

ST MARTIN'S PRESS INC
New York

PRINTED IN GREAT BRITAIN

PREFACE TO THE NEW AND REVISED EDITION OF 1910

THE whole book has been thoroughly revised, many alterations and additions have been made throughout, and a Chapter on Invariants has been added.

Additional illustrative examples have also been worked out in full.

An introduction to the subject of Envelopes is now given at the end of the Chapter on the Parabola.

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July 1, 1910

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[*The Articles marked with an asterisk may be omitted by beginners until after they have read Chapter IX.*]

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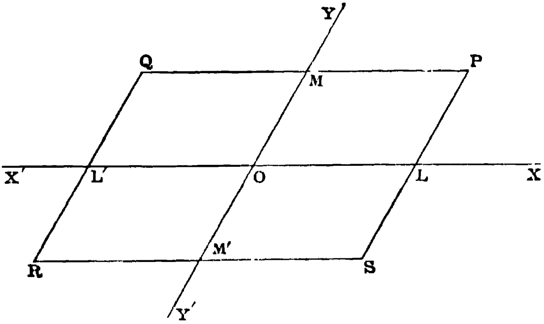
CHAPTER I.

CO-ORDINATES.

1. If in a plane two fixed straight lines XOX', YOY' be taken, and through any point P in the plane the two straight lines PM, PL be drawn parallel to XOX', YOY' respectively; the position of the point P can be found when the lengths of the lines PM, PL are given. For we have only to take OL, OM equal respectively to the known lines MP, LP and complete the parallelogram $LOMP$.



The lengths MP and LP, or OL and OM, which thus define the position of the point P with reference to the lines OX, OY are called the **co-ordinates** of the point P with reference to the **axes** OX, OY. The point of intersection of the **axes** is called the **origin**. When the angle between the axes is a right angle the axes are said to be **rectangular**; when the angle between the axes is not a right angle the axes are said to be **oblique**.

OL is generally called the **abscissa**, and LP the **ordinate** of the point P.

The co-ordinate which is measured along the axis OX is denoted by the letter x, and that measured along the axis OY by the letter y. If, in the figure, OL be a units of length and OM be b; then at the point P, $x = a$, and $y = b$, and the point is for shortness often called the point (a, b).

2. Let OM' be taken equal in length to OM, and OL' equal to OL, and through M', L' draw lines parallel to the axes, as in the figure to Art. 1. Then the co-ordinates of the three points Q, R, S will be equal in magnitude to those of P. Hence it is not sufficient to know the lengths of the lines OL, LP, we must also know the directions in which they are measured.

If lines measured in one direction be taken as positive, lines measured in the opposite direction must be taken as negative. We shall consider lines measured in the directions OX or OY to be positive, those therefore in the directions OX' or OY' must be considered negative.

We are now able to distinguish between the co-ordinates of the points P, Q, R, S. The co-ordinates of R are OL', $L'R$, and these are both measured in the negative direction; so that, if the co-ordinates of P be a, b, those of R will be $-a$, $-b$. The co-ordinates of S will be a, $-b$; and those of Q will be $-a$, b.

Thus P, Q, R, S are respectively the points (a, b), $(-a, b)$, $(-a, -b)$ and $(a, -b)$; also L, M, L', M' are the points $(a, 0)$, $(0, b)$, $(-a\ 0)$. $(0, -b)$.

When the co-ordinates of a point are supposed to be known they are generally represented, as in Algebra, by the earlier letters of the Alphabet—for example, (a, b), (c, d), (h, k), &c.—but when there is more than one known point the notation (x', y'), (x'', y''), &c. or (x_1, y_1), (x_2, y_2), &c. is generally used.

3. It must be carefully noticed that whether a line is positive or negative depends on the *direction* in which it is described, and does not depend on the position of the origin; for example, in the figure to Art. 1, the line LO is negative because the direction from L to O is the reverse of that from O to X.

If any two points K, L be taken and the distances OK, OL, measured from a point O in the line KL, be a and b respectively, then the distance KL must be $KO + OL$, or $-OK + OL$, that is $-a+b$, and this will be the case wherever the point O, from which distances are measured, may be.

If $OA=-3$, and $OB=4$; then

$$AB=AO+OB=-OA+OB=-(-3)+4=7.$$

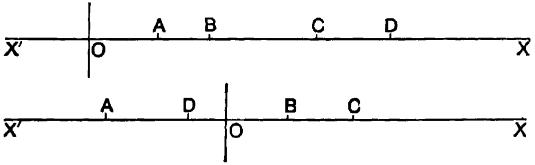
If $OA=3$, and $OB=-4$; then $AB=-3+(-4)=-7$.

The reader should illustrate this by means of a figure.

Ex. 1. If A, B, C, D be any four points in a straight line, then

$$AB \,.\, CD+BC \,.\, AD=AC \,.\, BD.$$

Take the straight line on which the points lie as the axis of x and any point O on it for origin. Then, if $OA=x_1$, $OB=x_2$, $OC=x_3$ and $OD=x_4$,



$$AB=AO+OB=-OA+OB=-x_1+x_2,$$
$$CD=CO+OD=-OC+OD=-x_3+x_4.$$

Also
$$BC=-x_2+x_3, \qquad AD=-x_1+x_4,$$
$$AC=-x_1+x_3 \text{ and } BD=-x_2+x_4.$$

Hence we have to prove that

$$(-x_1+x_2)(-x_3+x_4)+(-x_2+x_3)(-x_1+x_4)=(-x_1+x_3)(-x_2+x_4)$$

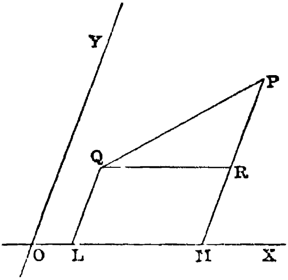
is true for all values of x_1, x_2, x_3, x_4; and this is easily seen.

Ex. 2. A, B, C are any three points on a straight line and P any other point. Prove that

$$PA^2 \,.\, BC+PB^2 \,.\, CA+PC^2 \,.\, AB+BC \,.\, CA \,.\, AB=0.$$

4. *To express the distance between two points in terms of their co-ordinates.*

Let P be the point (x', y'), and Q the point (x'', y''), and let the axes be inclined at an angle ω.



Draw PM, QL parallel to OY, and QR parallel to OX, as in the figure.

Then $OL = x''$, $LQ = y''$, $OM = x'$, $MP = y'$.

By trigonometry

$$PQ^2 = QR^2 + RP^2 - 2QR \, . \, RP \cos QRP.$$

But $$QR = LM = OM - OL = x' - x'',$$
$$RP = MP - MR = MP - LQ = y' - y'',$$
and angle QRP = angle $OMP = \pi$ − angle $XOY = \pi - \omega$,

$$\therefore \quad PQ^2 = (x' - x'')^2 + (y' - y'')^2 + 2(x' - x'')(y' - y'') \cos \omega,$$
or $PQ = \pm \sqrt{\{(x' - x'')^2 + (y' - y'')^2 + 2(x' - x'')(y' - y'') \cos \omega\}}$.

If the axes be at right angles to one another we have

$$PQ = \pm \sqrt{\{(x' - x'')^2 + (y' - y'')^2\}}.$$

The distance of P from the origin can be obtained independently, or by putting $x'' = 0$ and $y'' = 0$ in the above formula. The result is

$$OP = \pm \sqrt{\{x'^2 + y'^2 + 2x'y' \cos \omega\}},$$

or, if the axes be rectangular,

$$OP = \pm \sqrt{\{x'^2 + y'^2\}}.$$

Except in the case of straight lines parallel to one of the axes, no convention is made with regard to the direction which is to be considered positive. We may therefore suppose either PQ or QP to be positive. If however we have three or more points P, Q, R... in the same straight line, we must consider the same direction as positive throughout, so that in all cases we must have $PQ+QR=PR$.

In the following examples the axes are rectangular.

Ex. 1. Mark in a figure the position of the point $x=1$, $y=2$, and of the point $x=-3$, $y=-1$; and shew that the distance between them is 5.

Ex. 2. Find the lengths of the lines joining the following pairs of points: (i) $(1, -1)$ and $(-1, 1)$; (ii) $(a, -a)$ and $(-b, b)$; (iii) $(3, 4)$ and $(-1, 1)$.

Ex. 3. Shew that the three points $(1, 1)$, $(-1, -1)$ and $(-\sqrt{3}, \sqrt{3})$, are the angular points of an equilateral triangle.

Ex. 4. Shew that the four points $(0, -1)$, $(-2, 3)$, $(6, 7)$ and $(8, 3)$ are the angular points of a rectangle.

Ex. 5. Mark in a figure the positions of the points $(0, -1)$, $(2, 1)$, $(0, 3)$ and $(-2, 1)$, and shew that they are at the corners of a square.

Shew the same of the points $(2, 1)$, $(4, 3)$, $(2, 5)$ and $(0, 3)$.

Ex. 6. Shew that the four points $(2, 1)$, $(5, 4)$, $(4, 7)$ and $(1, 4)$ are the angular points of a parallelogram.

Ex. 7. If the point (x, y) be equidistant from the two points $(3, 4)$ and $(1, -2)$, then will $x+3y=5$.

[We have

$$(x-3)^2+(y-4)^2=(x-1)^2+(y+2)^2;$$

whence result.]

Ex. 8. Shew that the point $(7, 10)$ is equidistant from the three points $(-10, -9)$, $(32, 5)$ and $(18, 33)$.

Ex. 9. Find the point which is equidistant from $(0, 0)$, $(32, 10)$ and $(42, 0)$. *Ans.* $(21, -11)$.

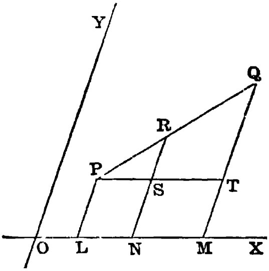
Ex. 10. Find the lengths of the sides of the triangle whose angular points are $(10, 8)$, $(13, 4)$ and $(\cdot 4, \cdot 8)$.

Shew that $(6\cdot 7, 2\cdot 4)$ is at a distance $6\cdot 5$ from each angular point.

Ans. The sides are 13, 12, 5.

5. *To find the co-ordinates of a point which divides in a given ratio the straight line joining two given points.*

Let the co-ordinates of P be x_1, y_1, and the co-ordinates of Q be x_2, y_2, and let $R(x, y)$ be the point which divides PQ in the ratio $k_1 : k_2$.



Draw PL, RN, QM parallel to the axis of y, and PST parallel to the axis of x, as in the figure.

Then $$LN : NM = PS : ST = PR : RQ = k_1 : k_2;$$

$$\therefore \quad k_2 . LN - k_1 . NM = 0,$$

or $$k_2(x - x_1) - k_1(x_2 - x) = 0;$$

$$x = \frac{k_2 x_1 + k_1 x_2}{k_1 + k_2}.$$

Similarly $$y = \frac{k_2 y_1 + k_1 y_2}{k_1 + k_2}.$$

The most useful case is when the line PQ is bisected: the co-ordinates of the point of bisection are

$$\tfrac{1}{2}(x_1 + x_2), \quad \tfrac{1}{2}(y_1 + y_2).$$

If the line were cut *externally* in the ratio $k_1 : -k_2$ we should have

$$LN : NM :: k_1 : -k_2,$$

and therefore $x = \dfrac{k_1 x_2 - k_2 x_1}{k_1 - k_2}$ $\quad y = \dfrac{k_1 y_2 - k_2 y_1}{k_1 - k_2}$.

The above results are true whatever the angle between the co-ordinate axes may be. But in most cases formulae become more complicated when the axes are not at right angles to one another.

We shall in future consider the axes to be at right angles in all cases except when the contrary is expressly stated.

Ex. 1. Find the middle point of the line joining (3, 1) and (-5, 7).

$$x=\tfrac{1}{2}\{3+(-5)\}=-1, \quad y=\tfrac{1}{2}(1+7)=4.$$

Ex. 2. Find the point which divides the join of (2, 3) and (5, -3) in the ratio 1 : 2.

$$x=\frac{2\times 2+5\times 1}{1+2}=3, \quad y=\frac{3\times 2+(-3)\times 1}{1+2}=1.$$

Ex. 3. The points A, B, C are respectively (x_1, y_1), (x_2, y_2) and (x_3, y_3). D, E, F are the middle points of BC, CA, AB respectively. Find the co-ordinates of the point G which divides DA so that $2DG=GA$.

[The co-ordinates of D are

$$x=\tfrac{1}{2}(x_2+x_3), \quad y=\tfrac{1}{2}(y_2+y_3).$$

Hence, for G, we have

$$x=\frac{\frac{1}{2}(x_2+x_3)\times 2+x_1\times 1}{1+2}=\tfrac{1}{3}(x_1+x_2+x_3),$$

$$y=\frac{\frac{1}{2}(y_2+y_3)\times 2+y_1\times 1}{1+2}=\tfrac{1}{3}(y_1+y_2+y_3).$$

The symmetry of this result proves that the point G is also on the lines EB and FC, and also that $2EG=GB$ and $2FG=GC$.

The intersection of the *medians* of a triangle is called the *centroid* of the triangle, and we see from the above that, if (x_1, y_1), (x_2, y_2), (x_3, y_3) are the angular points of a triangle, the co-ordinates of its centroid are

$$\tfrac{1}{3}(x_1+x_2+x_3) \text{ and } \tfrac{1}{3}(y_1+y_2+y_3).]$$

Ex. 4. Find the centroid of the triangle whose angular points are (-4, 6), (2, -2) and (2, 5) respectively. *Ans.* (0, 3).

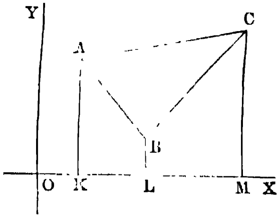
Ex. 5. Find the centroid of the triangle whose angular points are (3, -5), (-7, 4) and (10, -2) respectively. *Ans.* (2, -1).

Ex. 6. Find the point which divides the join of (5, -3) and (3, -5) in the ratio 3 : 5. *Ans.* $(\frac{17}{4}, -\frac{15}{4})$.

Ex. 7. Find the point which divides the join of (2, 1) and (3, 5) *externally* in the ratio 2 : -3. *Ans.* (0, -7).

6. *To express the area of a triangle in terms of the co-ordinates of its angular points.*

Let the co-ordinates of the angular points A, B, C be x_1, y_1; x_2, y_2; and x_3, y_3 respectively.



Draw the lines AK, BL, CM parallel to the axis of y, as in the figure.

$$\triangle ABC = MCAK - KABL - LBCM.$$

Now

$$\begin{aligned} MCAK &= \triangle MCA + \triangle AKM \\ &= \tfrac{1}{2} KM . MC + \tfrac{1}{2} KM . KA \\ &= \tfrac{1}{2}(x_3 - x_1)(y_3 + y_1). \end{aligned}$$

Similarly $KABL = \tfrac{1}{2}(x_2 - x_1)(y_1 + y_2)$,

and $LBCM = \tfrac{1}{2}(x_3 - x_2)(y_2 + y_3)$;

$$\therefore\ \triangle ABC = \tfrac{1}{2}\{(y_3 + y_1)(x_3 - x_1) + (y_1 + y_2)(x_1 - x_2) + (y_2 + y_3)(x_2 - x_3)\};$$

or, omitting the terms which cancel,

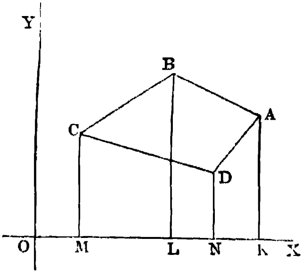
$$\triangle ABC = \tfrac{1}{2}\{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\}$$

$$= \tfrac{1}{2}\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}.$$

The above expression for the area of a triangle will be found to be positive if the order of the angular points be such that in going round the triangle the area is always on the left hand, or if the order of description of the circuit $ABCA$ is counterclockwise. Whenever on substitution a negative result for the area is obtained, a reverse order of proceeding round the triangle has been adopted.

7. *To express the area of a quadrilateral in terms of the co-ordinates of its angular points in a given order.*

Let the angular points A, B, C, D, taken in order, be (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4).



Draw AK, BL, CM, DN parallel to the axis of y, as in the figure.

Then the area $ABCD$

$$=KABL+LBCM-MCDN-NDAK.$$

And, as in the preceding Article,

$$\begin{aligned} KABL &= \tfrac{1}{2}(y_1+y_2)(x_1-x_2), \\ LBCM &= \tfrac{1}{2}(y_2+y_3)(x_2-x_3), \\ MCDN &= \tfrac{1}{2}(y_3+y_4)(x_4-x_3), \\ NDAK &= \tfrac{1}{2}(y_4+y_1)(x_1-x_4). \end{aligned}$$

Hence $ABCD = \frac{1}{2}\{(y_1+y_2)(x_1-x_2)+(y_2+y_3)(x_2-x_3)$
$+(y_3+y_4)(x_3-x_4)+(y_4+y_1)(x_4-x_1)\}$;

or, omitting the terms which cancel,

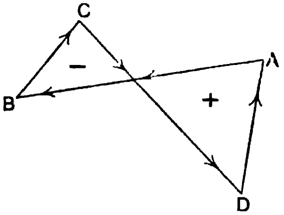
$$ABCD = \tfrac{1}{2}\{x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_4-x_4y_3+x_4y_1-x_1y_4\}.$$

The area of any polygon may be found in a similar manner.

[Another method is given in Art. 12.]

The above formula, beginning with $x_1y_2-x_2y_1$ and changing to $x_2y_3-x_3y_2$, &c. in cyclical order, is *positive* if the points are taken round the boundary of the figure in counterclockwise order, and is *negative* for the opposite direction.

It should however be noticed that four points can be joined in more than one way, and that in the figure below the formula would give the *difference* of the actual areas of the triangles marked + and − respectively.



Ex. 1. Find the area of the triangle whose angular points are (2, 1), (4, 3) and (2, 5).

Also find the area of the triangle whose angular points are (4, −5), (5, −6) and (3, 1). *Ans.* 4, $\frac{5}{2}$.

Ex. 2. Find the area of the triangle whose angular points A, B, C are respectively (2, 3), (1, 5) and (6, 2). *Ans.* −5.

[The negative sign shews that $ABCA$ is in the clockwise order of rotation, as will be seen if the points are plotted. In most cases only the absolute area is required.]

Ex. 3. A, B, C are the points (−1, 5), (3, 1) and (5, 7) respectively. D, E, F are the middle points of BC, CA, AB respectively. Prove

$$\triangle ABC = 4\triangle DEF.$$

Ex. 4. Find the area of the quadrilateral whose angular points taken in order are (1, 2), (6, 2), (5, 3) and (3, 4).

Also of the quadrilateral whose angular points are (2, 2), (−2, 3), (−3, −3) and (1, −2). *Ans.* $\frac{11}{2}$, 20.

Ex. 5. Find the area of the quadrilateral whose angular points A, B, C, D taken in order are (−4, 2), (3, −5), (1, 7) and (6, −2). *Ans.* 0.

Plot the points and draw $ABCDA$ to illustrate the result.

Find the area when the points are taken in the order A, B, D, C. *Ans.* 56.

Ex. 6. The points A, B, C, D are respectively (2, 4), (3, 2), (8, 4) and (7, 6). Find the area of $ABCD$. Also by taking the points in the order A, C, B, D and in the order A, B, D, C prove that AB is parallel to CD and BC to DA. *Ans.* 12.